

A TYPE OF CANTOR FUNCTIONS

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ABSTRACT. We construct a non-constant continuous function $f : [0, 1] \rightarrow \mathbb{R}$ which intersects the real line uncountable times and also it is not absolutely continuous. Functions of this type are very scarce in the literature.

1. INTRODUCTION

Note that we say a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ intersects the real line at $x = a$ if $f(a) = 0$ and in each neighborhood of $x = a$, there exist some points b and c such that $f(b) < 0$ and $f(c) > 0$. Recall that a function $f : [a, b] \rightarrow \mathbb{R}$ is said to be absolutely continuous if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon$ whenever $(a_1, b_1); (a_2, b_2) \dots; (a_n, b_n)$ are pairwise disjoint subintervals of $[a, b]$ for which $\sum_{i=1}^n |b_i - a_i| < \delta$.

The purpose of this note is to construct a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ which intersects the real line uncountable times and it is not absolutely continuous. To see the similar functions the reader is referred to [1, 2].

2. THE CONSTRUCTION

We construct a sequence of continuous functions on $[0, 1]$ such that the uniform limit of them is the desired function.

First, divide the interval $[0, 1]$ to three equal sections. Define \bar{f}_1 on $[0, \frac{1}{3}]$ as follows:

$$\bar{f}_1(x) = \begin{cases} 7x & 0 \leq x \leq \frac{1}{6} \\ -7x + \frac{7}{3} & \frac{1}{6} < x \leq \frac{1}{3} \end{cases}$$

It is easy to see that \bar{f}_1 is continuous and its graph is an isosceles triangle with the height $\frac{7}{6}$. Draw this triangle on the intervals $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ every other one, symmetrically. Let f_1 be the function on $[0, 1]$ whose graph is as above.

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From now on, our aim from lower (upper) triangle in each step is those triangles that their corresponding functions values are negative (positive), respectively. For construction of f_2 , fix the lower triangle drawn on the interval $[\frac{1}{3}, \frac{2}{3}]$ in the graph of f_1 and divide both intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ to three equal sections. Now, define the function \bar{f}_2 on the interval $[0, \frac{1}{9}]$ as follows:

$$\bar{f}_2(x) = \begin{cases} 10x & 0 \leq x \leq \frac{1}{18} \\ -10x + \frac{10}{9} & \frac{1}{18} < x \leq \frac{1}{9} \end{cases}$$

It is not difficult to verify that \bar{f}_2 is continuous and its graph is an isosceles triangle with the height $\frac{19}{18}$. Now, draw this triangle on the intervals $[\frac{1}{9}, \frac{2}{9}]$ and $[\frac{2}{9}, \frac{3}{9}]$ every other one, symmetrically. Then, copy the obtained graph on the interval $[0, \frac{1}{3}]$, on the interval $[\frac{2}{3}, 1]$ exactly and suppose that the obtained graph on the interval $[0, 1]$ is f_2 .

For the step 3, fix two lower triangles drawn on the intervals $[\frac{1}{9}, \frac{2}{9}]$ and $[\frac{7}{9}, \frac{8}{9}]$ and accomplish the same processes in the previous steps on the intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{3}{9}]$, $[\frac{6}{9}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$, respectively. Now, for the step n , define \bar{f}_n on the interval $[0, \frac{1}{3^n}]$ as follows:

$$\bar{f}_n(x) = \begin{cases} \frac{2(3^n)+n}{2n(3^n)}x & 0 \leq x \leq \frac{1}{2(3^n)} \\ -\frac{2(3^n)+n}{2n(3^n)}x + \frac{2(3^n)+n}{2n(3^{2n})} & \frac{1}{2(3^n)} < x \leq \frac{1}{3^n} \end{cases}$$

One can verify that \bar{f}_n is continuous and its graph is an isosceles triangle with the height $\frac{1}{n} + \frac{1}{2(3^n)}$ and then do the same tasks in the previous steps. In fact in this step we have 2^n upper isosceles triangles with the height $\frac{1}{n} + \frac{1}{2(3^n)}$ and $2^n - 1$ lower isosceles triangles which 2^{n-1} of them have the height $\frac{1}{n} + \frac{1}{2(3^n)}$. Let $\varepsilon > 0$ be arbitrary. Then $\|f_n - f_m\|_\infty \leq \frac{1}{n} + \frac{1}{2(3^n)} + \frac{1}{m} + \frac{1}{2(3^m)} < \varepsilon$, for sufficiently large $m, n \in \mathbb{N}$, in which $\|\cdot\|_\infty$ means the supremum norm. So, by using the Cauchy test for uniform convergence (See [1]), (f_n) converges uniformly to a continuous functions f . Therefore, f is a limit function which is constructed according to the construction of Cantor set and thus it is identically to zero on the whole of Cantor set and by the process of the construction, f intersects the real line uncountable times. Now, we show that f is not absolutely continuous. Let $\delta > 0$ be arbitrary. There exists $k \in \mathbb{N}$ such that $\frac{1}{2(3^k)} < \delta$. Put $\bar{\delta} = \frac{1}{2(3^k)}$ and consider the sequence $(\frac{\bar{\delta}}{3^n})_{n \in \mathbb{N}}$. From 2^{p-1} lower triangles constructed in the step p with the height $\frac{1}{p} + \frac{1}{2(3^p)}$ ($p = k, k+1, \dots$), choose one of them and consider a neighborhood from the initial of the base of the triangle to the its middle on the real line. We have $\sum_{n=1}^{\infty} \frac{\bar{\delta}}{3^n} = \frac{\bar{\delta}}{2} < \delta$.

Let $m > k$ be arbitrary. Then, $\sum_{i=k}^m |f(b_i) - f(a_i)| > \sum_{i=k}^m \frac{1}{i}$, in which a_i and b_i are the points of the initial and the middle of the base of the chosen lower triangle in the step i , respectively and we have $f(a_i) = 0$ and $f(b_i) > \frac{1}{i}$. This shows that f can not be absolutely continuous and so f has the desired properties.

REFERENCES

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